Section 10.6

Alternating Series Test

The series of interest in this section will take the form

$$\sum_{k=m}^{\infty}(-1)^{k+1}a_k \text{ where } a_k \geq 0,$$

the *alternating series* so named because the terms alternate in sign.

Theorem 10.6.1. Alternating Series Test
The alternating series $\sum(-1)^{k+1}a_k$ converges provided that

1. the terms of the series are non-increasing in magnitude: $0 \leq a_{k+1} < a_k$ for k greater than some N.
2. $\lim_{k\to\infty} a_k = 0$.

There are two things to note. First, unlike the divergence test $\lim_{k\to\infty} a_k = 0$ implies convergence for alternating series. Second, none of the other convergence tests from previous sections apply to alternating series!

Example 10.6.2.

1. The Alternating Harmonic Series: $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$

2. $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k \ln k}{k}$

3. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$

Definition 10.6.3.
If $\sum |a_k|$ converges, then $\sum a_k$ *converges absolutely.*

If $\sum |a_k|$ diverges and $\sum a_k$ converges, then $\sum a_k$ *converges conditionally.*

So $\sum \frac{(-1)^{k+1}}{k^2}$ is absolutely convergent, while $\sum \frac{(-1)^{k+1}}{k}$ converges conditionally.

Theorem 10.6.4. Absolute Convergence Implies Convergence

If $\sum |a_k|$ converges, then $\sum a_k$ converges. If $\sum a_k$ diverges, then $\sum |a_k|$ diverges.

Example 10.6.5. Determine whether the following series converge. If the series converges, determine if it converges absolutely or conditionally.

1. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$

2. $\displaystyle\sum_{k=1}^{\infty} (-1)^k \left(1 + \frac{1}{k}\right)^k$

3. $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(\pi k)}{k^2}$

4. $\displaystyle\sum_{k=1}^{\infty} (-1)^k k \sin\left(\frac{1}{k}\right)$

5. $\displaystyle\sum_{k=3}^{\infty} \frac{(-1)^k}{\ln k}$

6. $\displaystyle\sum_{k=1}^{\infty} \frac{\cos k}{k^3}$

7. $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{3^k - 1}$

8. $\displaystyle\sum_{k=2}^{\infty} (-1)^k \frac{k^2 + 1}{k^3 - 1}$

9. $\displaystyle\sum_{k=1}^{\infty} \frac{\sin k}{3^k + 4^k}$

10. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k!}{k^k}$